CS 237: Probability in Computing

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Lecture 4:

- Conditional Probability
- Independent vs Dependent Events
- Bayes' Rule
- Law of Total Probability
- The Base Rate Fallacy

Review: Tree Diagrams for Solving Probability Problems

Problem: Wayne is buying CDs, and the probability that he buys Classical music is 0.5, Rock is 0.3, and Blues is 0.2. He buys 2 CDs.

Let A = "the first CD he buys is Classical"

B = "he buys 2 different genres."

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Now let us consider:

The first CD he buys is Classical. Now what is P(B)?

OR: What is the probability that he buys 2 different genres, given that the first CD he buys is classical?

OR: What is the probability of B, given that we know A is true?

Review: Tree Diagrams for Solving Probability Problems

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Conditional Probability: P(A | B)

How does new information B affect the probability of an event A?

P(A | B) = The probability that A occurs, given that B HAS occurred.

Example: Suppose I toss two dice, one in front of you, and the other where you can not see the result. The one you can see shows 3 dots. What is the probability that more than 8 dots showed one both dice?

Or: P(A | B)

where:

A = "The total number of dots is more than 8"

B = "The first die shows 3 dots"

Conditional Probability: P(A | B)

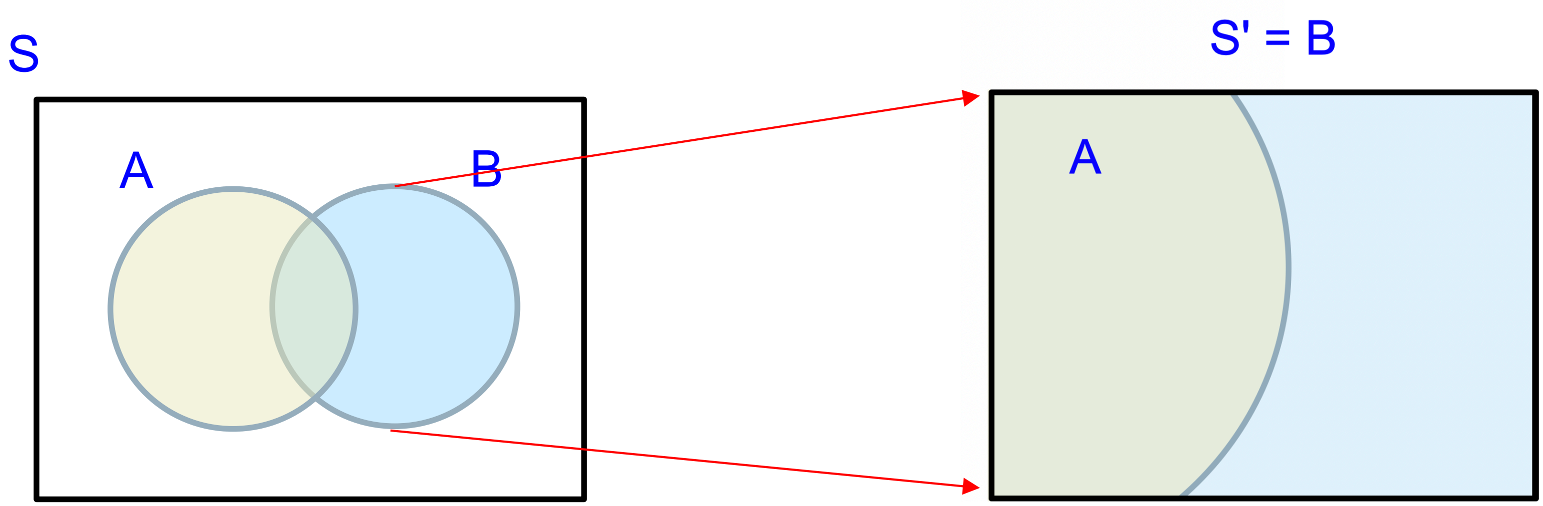
Example: Roll two dice. A = "the total dots is > 8" and B = "the first roll was 3"
What is P(A | B)?

The key to solving such problems is to realize that there are two probability spaces:

- the one before you know whether B has happened, and
- the one that has been "conditioned" by knowing that B has definitely happened, so the sample space has shrunk and the proportion representing event A may have changed:

Original

Conditioned by knowing B happened:



Conditional Probability: P(A | B)

Example: Roll two dice. A = "the total # dots is > 8" and B = "the first roll was 3"
What is P(A | B)?

Original

P(A) = ?

P(B) = ?

P(A ∩ B) =

Second Roll

First Roll	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

A

First Roll	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

B

Conditional Probability: P(A | B)

Example: Roll two dice. A = "the total # dots is > 8" and B = "the first roll was 3"
What is P(A | B)?

Original

First Roll

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

A

Conditioned by knowing B happened

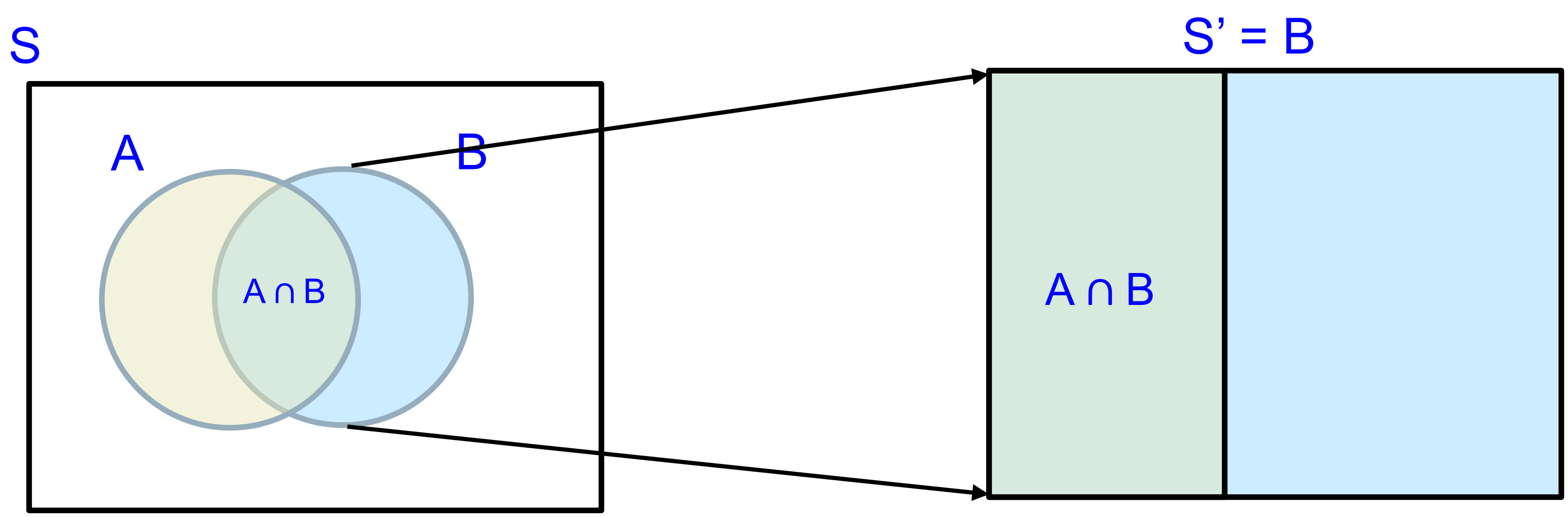
3	4	5	6	7	8	9

P(A | B) = 1/6

Conditional Probability

Conditioning the original sample space means changing the perspective: instead of finding the area of A inside S, we are finding the area of $A \cap B$ inside B:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$



Conditional Probability: P(A | B)

Example: Roll two dice. A = "the total # dots is > 8" and B = "the first roll was 3"
What is P(B | A)?

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

Original

Second Roll

First Roll	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

B

A

Conditioned by knowing A happened:

Second Roll

First Roll	1	2	3	4	5	6
1						
2						
3						9
4					9	10
5				9	10	11
6			9	10	11	12

P(B | A) = 1/10

Conditional Probability: $P(A \mid B)$

Example: A family has two children, an older and a younger. Both genders are equally likely.

(A) What is the probability that both children are girls, given that the younger is a girl?

(B) What is the probability that both children are girls, given that at least one of the children is a girl?

Conditional Probability: P(A | B)

Example: A family has two children, an older and a younger. Both genders are equally likely.

(A) What is the probability that both children are girls, given that the younger is a girl?

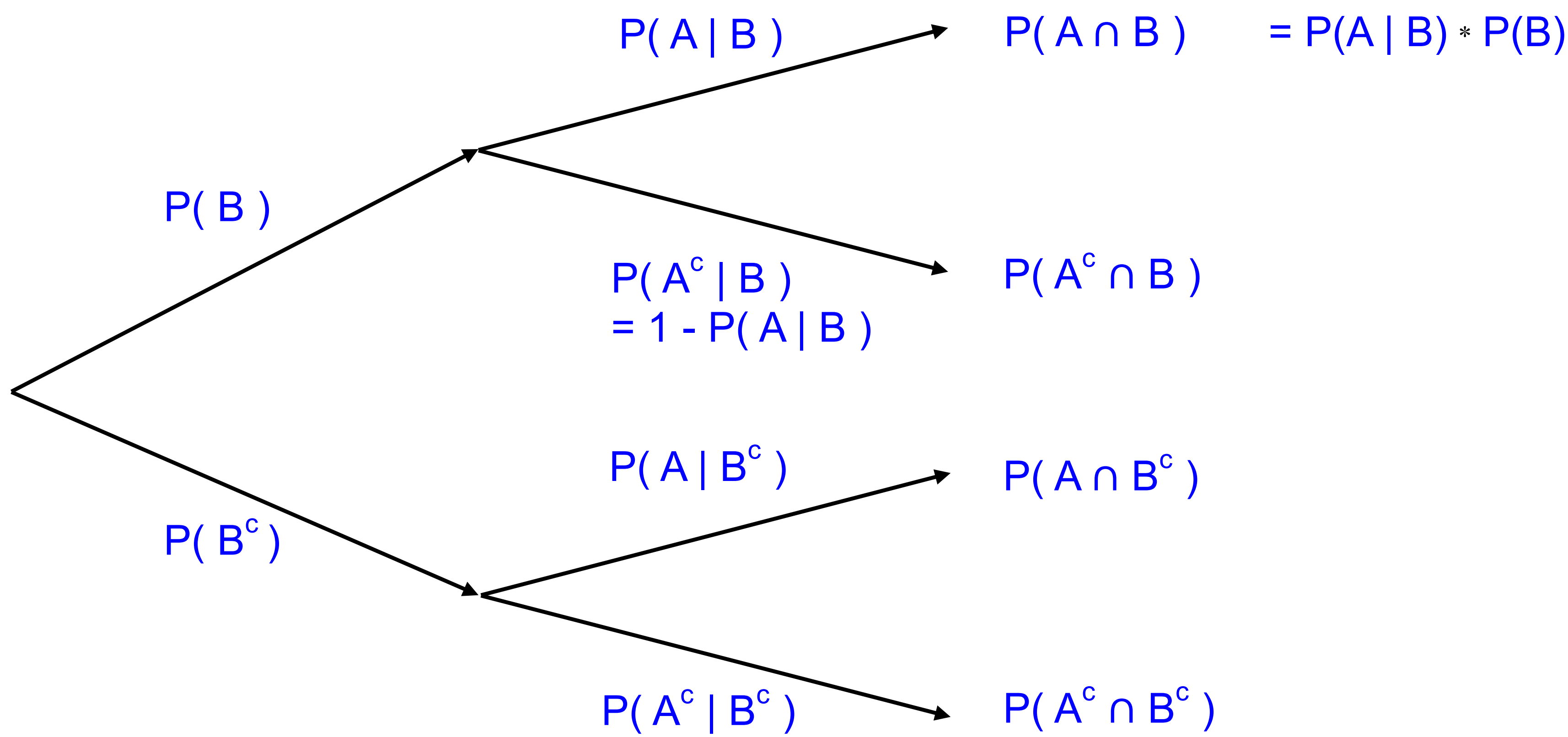
(B) What is the probability that both children are girls, given that at least one of the children is a girl?

Conditional Probability and Tree Diagrams

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

P(A | B) considers an event B followed by an event A, and how the occurence of B affects the occurence of A. What are the labels on a tree diagram of this random experiment?

B occurs (or not) A occurs (or not)



Conditional Probability and Tree Diagrams

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Independence and Dependence

We say that two events A and B are independent if

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \mid B) = P(A)$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

or, equivalently, and most importantly as we go forward:

$$P(A \cap B) = P(A) * P(B)$$

If two events are NOT independent, then they are dependent.

Example:

What is the probability of getting HHT when flipping three fair coins?

P(HHT) = P(H) * P(H) * P(T) = ½ * ½ * ½ = 1/8.

Note: Independence does not depend on physical independence, and dependence does not imply a causal relationship. However, it gives you some evidence!

Independence and Dependence

We say that two events A and B are independent if

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$$P(A \mid B) = P(A)$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

or:

$$P(A \cap B) = P(A) * P(B)$$

Example:

Suppose in a particular city, 40% of the population is male, and 60% female, and 20% of the population smokes. If male smokers are 8% of the population, then are smoking and gender independent? That is, are the following two events independent?

A = Smoker

B = Male

Independence and Dependence

We say that two events A and B are independent if

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \mid B) = P(A)$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

or:

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Example:

Suppose in a particular city, 40% of the population is male, and 60% female, and 20% of the population smokes. If male smokers are 8% of the population, then are smoking and gender independent? That is, are the following two events independent?

YES. Check:

A = Smoker

$P(A \cap B) = 0.08 = 0.4 * 0.2 = P(A) * P(B)$

B = Male

Independence and Dependence

Digression: Dependence does not imply causality!

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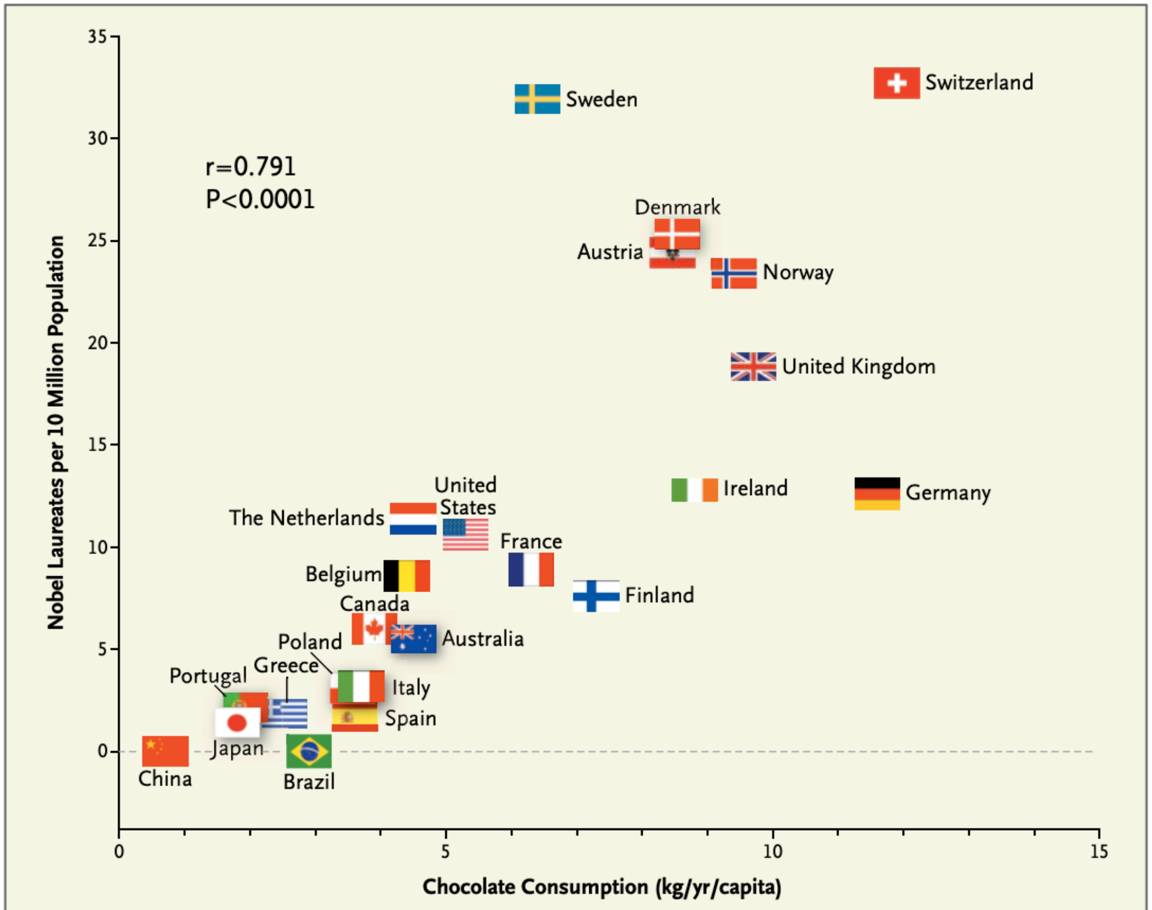

Figure 1. **Correlation between Countries' Annual Per Capita Chocolate Consumption and the Number of Nobel Laureates per 10 Million Population.**

Independence and Dependence

How does this relate to tree diagrams?

When the events are independent, then we have the familiar tree diagram in which we simply write the probabilities of the events on each arc: